# Chapter 3

# LONGITUDINAL PHASE SPACE

## 3.1 MOMENTUM COMPACTION

A bunch of charged particles has a spread of energy because of many reasons, for example, random quantum excitation which changes the energy of the particles randomly (for electrons only), intrabeam scattering which is just Coulomb scattering among the particles, Touschek scattering [1] which is large-angle Coulomb scattering which converts the transverse momentum of a particle into longitudinal, and last and most important of all a means to counter collective instabilities through Landau damping. In an accelerator ring or storage ring, particles with different energies have different closed orbits, their lengths are given by

$$C = C_0 \left[1 + \alpha_0 \delta + \mathcal{O}(\delta^2)\right] \ , \tag{3.1}$$

where $\delta$ is the fractional spread in momentum and $C_0$ is the orbit length of the so-called *on-momentum* particle. The proportionality constant $\alpha_0$ is called the momentum-compaction factor of the accelerator ring. The fraction momentum spread is related to the fraction energy spread $\Delta E/E_0$ by

$$\delta = \frac{\Delta p}{p_0} = \frac{1}{\beta^2}\frac{\Delta E}{E_0} \ . \tag{3.2}$$

where $p_0$ and $E_0$ are the momentum and energy of the on-momentum particle. The momentum-compaction factor of most accelerator and storage rings have $\alpha_0 > 0$, implying that particles with larger energy will travel along longer closed orbits with more radial excursions. Thus the period of revolution will be relatively longer. Therefore, particles with lower energies will slip ahead by the time $\Delta T$ every turn, while particles with higher energies will slip behind. The particles will spread out longitudinally and the bunch will spread out and disintegrate. The *slip factor* $\eta$ is defined as

$$\eta = \frac{\Delta T}{T_0} = \frac{\Delta C}{C_0} - \frac{\Delta v}{v_0} = \alpha_0 - \frac{1}{\gamma^2} \ , \tag{3.3}$$

where $T_0$ and $v_0$ are, respectively, the revolution period and velocity of the on-momentum particle, and $\gamma = E_0/m$, $m$ being the particle rest mass. For most electron rings and high energy proton rings, the particle velocity $v$ is extremely close to $c$, the velocity of light. We therefore have actually $\eta \approx \alpha_0$ and we called the operation *above the transition energy.* For low-energy hadron rings, the velocity term may dominate making $\eta < 0$ and we say the operation is *below the transition energy.* Obviously the transition energy is defined as $E_t = \gamma_t mc^2$ and $\gamma_t = \alpha_0^{-1/2}$. There are also rings, like the 1.2 GeV antiproton ring LEAR at CERN and many newly designed ones [2] that have negative momentum-compaction factors or $\alpha_0 < 0$. In these rings, lower momentum particles have longer closed orbits or larger radial excursions than higher momentum particles. Negative momentum compaction implies an imaginary $\gamma_t$ and the slip factor will always be negative, indicating that the ring will be always below transition. Some believe that such rings will be more stable against collective instabilities [3]. Design and study of negative momentum compaction rings have been an active branch of research in accelerator physics lately [4].

In order to have the particles bunched, a longitudinal focusing force will be required. This is done by the introduction of rf cavities. Consider 3 particles arriving in the first turn at exactly the same time at a cavity gap, where the rf sinusoidal voltage curve is at 180°, as shown in Fig. 3.1a. All three particles are seeing zero rf voltage and are not gaining any energy from the rf wave. Assume the ring is above transition or $\eta > 0$. One turn later, the on-momentum particle arrives at the cavity gap at exactly the time when the rf sinusoidal voltage curve is again at 180° and gains no energy. The lower energy particle arrives at the gap earlier by $\tau_1$, which we call *time slip*. It sees the positive part of the rf voltage and gains energy, as illustrated in Fig. 3.1b. For the second turn, it will arrives at the gap earlier by $\tau_1 + \tau_2$, where $\tau_2 < \tau_1$ because the particle energy has been raised in the second passage. This particle will continue to gain energy from the rf every turn and its turn-by-turn additional time slip diminishes. Eventually, this particle will have an energy higher than the on-momentum particle and starts to arrive at the cavity gap later turn after turn, or its turn-by-turn time slip becomes negative. Similar conclusion can be drawn for the particle that has initial energy higher than the on-momentum particle. With the rf voltage wave, the off-momentum particles will oscillate around the on-momentum particle and continue to form a bunch. In reality, the particles lose an amount of energy

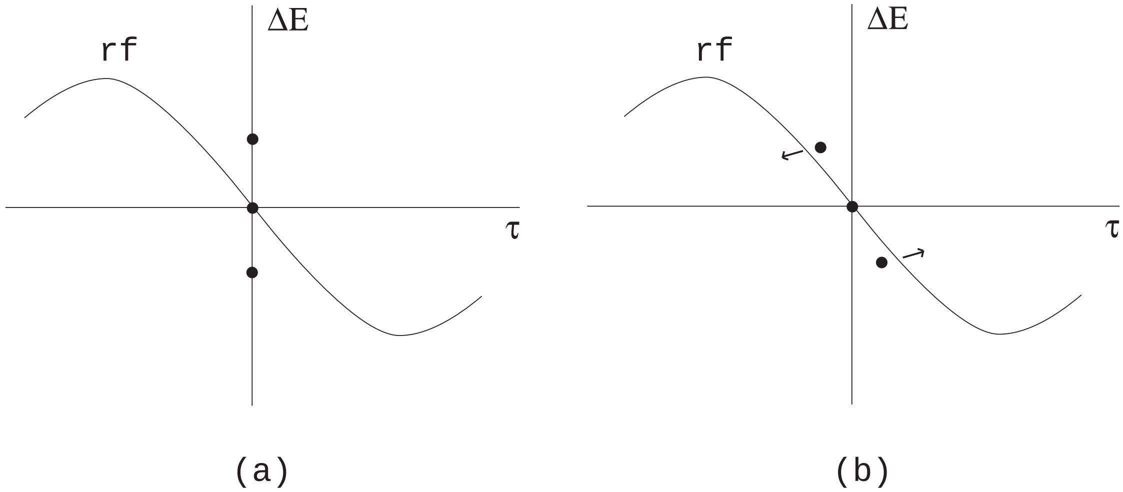


Figure 3.1: Three particles are shown in the longitudinal phase planes. (a) Initially, there are all at the rf phase of 180° and do not gain or lose any energy. (b) One turn later, the on-momentum particle arrives with the same phase of 180° without any change in energy. The particle with lower energy arrives earlier and gains energy from the positive part of the rf at phase $< 180°$. The particle with higher energy arrives late and loses energy because it sees the rf at phase $> 180°$.

$U_s$ every turn due to synchrotron radiation. This is compensated by shifting the rf phase slightly from 180° to $\phi_s = \sin^{-1}(U_s/V_{\rm rf})$ so that the on-momentum electron will see the rf voltage at the phase $\phi_s$ when traversing the cavity gap. This particle is also called the *synchronous* particle.

## 3.2 EQUATIONS OF MOTION

To measure the charge distribution in a bunch, we choose a fixed reference point $s_0$ along the ring and put a detector there. A particle in a bunch is characterized longitudinally by $\tau$, the time it arrives at $s_0$ ahead of the synchronous particle. We record the amount of charge arriving when the time advance is between $\tau$ and $\tau + d\tau$. The result is $e\rho(\tau)d\tau$, where $\rho(\tau)$ is a measure of the particle distribution and $e$ is the particle charge. The actual linear particle density per unit length is $\lambda(\tau) = \rho(\tau)/v$, where $v$ is the velocity of the synchronous particle. Note that this charge distribution is measured at a fixed point but at different times. Therefore, it is *not* a periodic function of $\tau$. In one turn, the change in time advance is

$$\Delta\tau = -\eta T_0 \delta \ . \tag{3.4}$$

The negative sign comes about because the period of a higher-momentum particle is larger above transition ($\eta > 0$) and therefore its time of arrival slips. During that turn, the energy gained by the particle relative to the synchronous particle is

$$\Delta E = eV_{\rm rf}(\sin\phi - \sin\phi_s) - [U(\delta) - U_s] + C(\langle F_0^\parallel\rangle - \langle F_{0s}^\parallel\rangle) \ , \tag{3.5}$$

where the subscript $s$ stands for synchronous particle, and $C$ is the ring circumference. Note that as a first approximation, we do not distinguish between $C$ and $C_0$. The first term on the right is the sinusoidal rf voltage and the second term is the radiation energy. The third is the wake force defined in the previous section due to all beam particles ahead; it can therefore be written as

$$\langle F_0^\parallel(\tau)\rangle = -\frac{e^2}{C}\int_0^\infty d\tau'\rho(\tau')W_0'(\tau'-\tau) \ . \tag{3.6}$$

Notice that we have written, for convenience, the wake function as a function of time advance. The $\langle F_{0s}^\parallel\rangle$ is the wake force on the synchronous particle. It is a constant energy loss, which is compensated by suitably choosing the synchronous phase $\phi_s$.

The two equations of motion are related because the momentum spread is related to the energy spread by $\delta = \Delta E/(\beta^2 E_0)$, and the rf phase seen is related to the time advance,

$$\phi - \phi_s = -h\omega_0\tau \ , \tag{3.7}$$

where $\omega_0/(2\pi) = 1/T_0$ is the revolution frequency of the ring, $V_{\rm rf}$ is the rf voltage (the peak value of the rf wave), and $h$ is the rf harmonic, which is the number of oscillations the rf wave makes during one revolution period. Here, $q$ is absolute value of the charge of the beam particle in the ring; we often write $q = e$ because we are dealing mostly with protons or electrons. The negative sign on the right-hand side of Eq. (3.7) comes about because when the particle arrives earlier or $\tau > 0$, it sees a rf phase *earlier* than the synchronous phase $\phi_s$. Writing as *discrete* differential equations, they become

$$\frac{d\tau}{dn} = -\frac{\eta T_0}{\beta^2}\frac{\Delta E}{E_0} \ , \tag{3.8}$$

$$\frac{d\Delta E}{dn} = eV_{\rm rf}[\sin(\phi_s - h\omega_0\tau) - \sin\phi_s] - [U(\delta) - U_s] + C(\langle F_0^\parallel\rangle + \langle F_{0s}^\parallel\rangle) \ . \tag{3.9}$$

To simplify the mathematics, a continuous independent variable is needed instead of the discrete turn number. Time is not a good variable here because it is complicated by synchrotron motion and the acceleration process. We choose instead $s$, the distance along the closed orbit of the synchronous particle. With $\tau$ and $\Delta E$ as the canonical variables*, the equations of motion for a particle in a small bunch become

$$\frac{d\tau}{ds} = -\frac{\eta}{v\beta^2 E_0}\,\Delta E\ , \tag{3.10}$$

$$\frac{d\Delta E}{ds} = \frac{eV_{\rm rf}}{C}[\sin(\phi_s - h\omega_0\tau) - \sin\phi_s] - \frac{U - U_s}{C} + \langle F_0^{\parallel}\rangle\ . \tag{3.11}$$

In the absence of the wake potential, if we neglect the small difference between the energy lost $U(\delta)$ by the off-momentum particle and the energy lost $U_s$ by the on-momentum particle. for small amplitude oscillations, the two equations combine to give

$$\frac{d^2\tau}{ds^2} - \frac{2\pi\eta h eV_{\rm rf}\cos\phi_s}{C^2\beta^2E_0}\,\tau = 0\ . \tag{3.12}$$

Therefore, the bunch particles are oscillating with the angular frequency $\omega_{0s} = \nu_{s0}\omega_0$, where

$$\nu_{0s} = \sqrt{-\frac{\eta heV_{\rm rf}\cos\phi_s}{2\pi\beta^2E_0}}\ , \tag{3.13}$$

is called the *synchrotron tune* and $\omega_{0s}/(2\pi)$ the synchrotron frequency. The subscript "0" indicates that these are the *unperturbed* small-amplitude values or with the wake potential turned off. The negative sign inside the square root implies that $\phi_s$ should be near 180° in the second quadrant above transition ($\eta > 0$), but near 0° in the first quadrant below transition ($\eta < 0$). When the oscillation amplitude is larger, the sine wave cannot be linearized. The focusing force is smaller and the synchrotron tune will become smaller. In other words, there will be a spread in the synchrotron tune which will be very essential to the Landau damping of the collective instabilities to be discussed later. As the oscillation amplitude continues to increase, a point will be reached when there is no more focusing provided anymore. This boundary in the $\tau$-$\Delta E$ phase space gives the maximum possible bunch area allowed and is called the *bucket* holding the bunch. Any particle that goes outside the bucket will be lost.

If the radiation energy is neglected, the two equations of motion are derivable from the Hamiltonian

$$H = -\frac{\eta}{2v\beta^2E_0}(\Delta E)^2 - \frac{eV_{\rm rf}}{Ch\omega_0}\Big[\cos(\phi_s - h\omega_0\tau) - \cos\phi_s - h\omega_0\tau\sin\phi_s\Big] + V(\tau)\ , \tag{3.14}$$

*This set of canonical variables should not be used if the accelerator is ramping.

with the aid of the Hamiltonian equations

$$\begin{cases} \dfrac{d\tau}{ds} = \dfrac{\partial H}{\partial \Delta E} \ , \\ \dfrac{d\Delta E}{ds} = -\dfrac{\partial H}{\partial \tau} \ . \end{cases} \tag{3.15}$$

Here,

$$V(\tau) = \frac{e^2}{C} \int_0^\tau d\tau'' \int_{\tau''}^\infty d\tau' \rho(\tau') W_0'(\tau' - \tau'') \ . \tag{3.16}$$

For small amplitude oscillations, the Hamiltonian simplifies to

$$H = -\frac{\eta}{2v\beta^2 E_0}(\Delta E)^2 - \frac{\omega_{0s}^2 \beta^2 E_0}{2\eta v}\tau^2 + V(\tau) \ . \tag{3.17}$$

In an electron ring, synchrotron radiation may provide damping to many collective instabilities. Because this damping force is dissipative in nature, strictly speaking a Hamiltonian formalism does not apply. However, the synchrotron radiation damping time is usually very much longer than the synchrotron period. The fast growing instabilities will evolve to their full extent before the damping mechanism sets in. Here, we are interested mostly in studying those instabilities that grow within one radiation damping time of the ring. For a time period much less than the radiation damping time, radiation can be neglected and the Hamiltonian formalism therefore applies.

## 3.3 VLASOV EQUATION

We would like to study the evolution of a bunch that contains, say, $10^{12}$ particles. The Hamiltonian in Eq. (3.14) has to be modified to include $10^{12}$ sets of canonical variables in order to fully describe the bunch. The description of the motion of a collection of $10^{12}$ particles is known as the *particle approach*, and is often tackled in the time domain. However, what are of interest to us are the collective behavior of the bunch like the motion of its centroid, the evolution of the particle distribution, etc. In other words, we are studying here the evolution of various modes of motion of these collective variables. For $10^{12}$ particles, there are $10^{12}$ modes of motion. However, we will never be interested in those modes whose wavelengths are of the magnitude of the separation between two adjacent particles inside the bunch, because they will correspond to motions of very high frequencies, and those motions are microscopic in nature. What we would like to study are the macroscopic modes of the bunch, or those having wavelengths of the same order as the length of the bunch or the radius of the vacuum chamber. Sometimes, we may even want to study modes with wavelengths one tenth of one hundredth of the bunch length or beam pipe radius, but definitely not down to the microscopic size like the particle oscillation. In other words, we go to the frequency domain and look at the different modes of motion of oscillation of the bunch, our interest is on those few modes that have the lowest frequencies or longest wavelengths. This direction of study is known as the *mode approach*.

When collisions are neglected, the basic mathematical tool for the mode approach is the Vlasov equation or the Liouville theorem [5]. It states that if we follow the motion of a representative particle in the longitudinal or $\tau$-$\Delta E$ phase space, the density of particles in its neighborhood is constant. In other words, the distribution of particles $\psi(\tau,\Delta E;s)$ moves in the longitudinal phase space like an incompressible fluid. Mathematically, the Vlasov equation reads

$$\frac{d\psi}{ds}=\frac{\partial\psi}{\partial s}+\frac{d\tau}{ds}\frac{\partial\psi}{\partial\tau}+\frac{d\Delta E}{ds}\frac{\partial\psi}{\partial\Delta E}=0\ . \tag{3.18}$$

In terms of the Hamiltonian, it becomes

$$\frac{\partial\psi}{\partial s}+[\psi,H]=0\ , \tag{3.19}$$

where $[,]$ denotes the Poisson bracket. Here, the time of early arrival $\tau$ and the energy offset $\Delta E$ are the set of canonical variables chosen. The Poisson bracket is therefore

$$[\psi,H]=\frac{\partial\psi}{\partial\tau}\frac{\partial H}{\partial\Delta E}-\frac{\partial\psi}{\partial\Delta E}\frac{\partial H}{\partial\tau}\ . \tag{3.20}$$

Together with the Hamiltonian equations of Eq. (3.15), Eq. (3.18) is reproduced.

## 3.4 EXERCISES

3.1. The Hamiltonian of Eq. (3.14) describes motion in the longitudinal phase space, when the wake potential $V(\tau)$ is not included. With the effects of the wake potential neglected, find the fixed points of the Hamiltonian above and below transition, and determine whether they are stable or not. The separatrices are the contours of fixed Hamiltonian values that pass through the unstable fixed points. They separate the region of libration motion (oscillatory motion) from rotation motion. Plot the separatrices.

3.2. The canonical variables $\tau_0$ and $\Delta E_0$ evaluated at 'time' $s = 0$ become $\tau_1$ and $\Delta E_1$ at an infinitesimal time $\Delta s$ later according to

$$\tau_1 = \tau_0 + \frac{\partial H}{\partial \Delta E_0}\Delta s\,, \qquad \Delta E_1 = \Delta E_0 - \frac{\partial H}{\partial \tau_0}\Delta s\,. \tag{3.21}$$

Consider the small phase-space area element $d\tau_0 d\Delta E_0 = J d\tau_1 d\Delta E_1$. Show that the Jacobian $J = 1$ to the first order of $\Delta s$, implying that the area surrounding a given number of particles does not change in time, which is Liouville Theorem. It is possible to prove $J = 1$ to all orders of $\Delta s$ using canonical transformation. See, for example, H. Goldstein, *Classical Mechanics*, Addison-Wesley, Chapter 8-3.

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